MATH 105 101 Midterm 1 Sample 1

1. (20 marks)

(a) (5 marks) Let

$$f(x, y) = \arctan(xy)$$

Compute $f_{xx}(1, 1)$ and $f_{xy}(1, 1)$. *Simplify your answer.*

Solution: Calculate the first-order partial derivative of f with respect to x:

$$f_x(x, y) = \frac{y}{1 + x^2y^2}.$$

Calculate $f_{xx}(x, y)$ and $f_{xy}(x, y)$:

$$f_{xx}(x, y) = \frac{-2xy^3}{(1 + x^2y^2)^2}, \qquad f_{xy}(x, y) = \frac{(1 + x^2y^2) - y(2x^2y)}{(1 + x^2y^2)^2} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}.$$

Evaluate f_{xx} and f_{xy} at $(1, 1)$:

$$f_{xx}(1, 1) = \frac{-2}{4} = -\frac{1}{2}, \qquad f_{xy}(x, y) = \frac{0}{4} = 0.$$

Thus, $f_{xx}(1, 1) = -\frac{1}{2}$, and $f_{xy}(1, 1) = 0$.

(b) (5 marks) Find all vectors in $\mathbb{R}^3$ of length 10 which are parallel to the vector $\langle 3, 0, -4\rangle$.

Solution: Let $\mathbf{v}$ be such a vector. Then, since $\mathbf{v}$ is parallel to $\langle 3, 0, -4\rangle$, we have that $\mathbf{v} = c\langle 3, 0, -4\rangle = \langle 3c, 0, -4c\rangle$ for some real constant c. Since $\mathbf{v}$ has length 10, we get:

$$\begin{aligned} 10 &= \sqrt{(3c)^2 + 0^2 + (-4c)^2} = \sqrt{25c^2} = 5|c| \\ 2 &= |c| \\ \Rightarrow c &= \pm 2 \end{aligned}$$

Thus, there are two vectors in $\mathbb{R}^3$ of length 10 which are parallel to the vector $\langle 3, 0, -4\rangle$: $\langle 6, 0, -8\rangle$ and $\langle -6, 0, 8\rangle$.

(c) (2 marks) Find an equation of the plane $\mathcal{P}$ passing through the point $(3, -1, 4)$ that is parallel to the plane $3x - 5y = 3$.

Solution: Since $\mathcal{P}$ is parallel to the plane $3x-5y=3$, $\mathcal{P}$ has the same normal vector as that plane, which is $\langle 3,-5,0\rangle$. So, an equation of the plane $\mathcal{P}$ is:

$$\begin{aligned} 3(x-3)-5(y+1)+0(z-4)&=0 \\ \Rightarrow 3x-5y&=14 \end{aligned}$$

(d) (3 marks) Is there a function $f(x,y)$ such that $\nabla f(x,y)=\langle \cos y, x\sin y+y^2\rangle$? If not, explain why no such function exists; otherwise find $f(x,y)$. State clearly any result that you use.

Solution: Suppose that $f(x,y)$ is a function such that $\nabla f(x,y)=\langle \cos y, x\sin y+y^2\rangle$. Then,

$$f_{xy}=-\sin y\neq \sin y=f_{yx}.$$

Note that f_{xy} and f_{yx} are both continuous on $\mathbb{R}^2$, being trigonometric functions. So, $f_{yx}\neq f_{xy}$ contradicts Clairaut's Theorem which states that if f_{yx} and f_{xy} are continuous, then $f_{xy}=f_{yx}$. Therefore, there does not exist a function $f(x,y)$ such that $\nabla f(x,y)=\langle \cos y, x\sin y+y^2\rangle$.

(e) (5 marks) Consider the surface S given by:

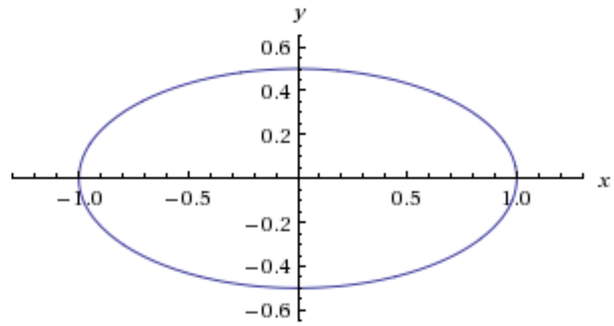
$$z^2=x^2+4y^2$$

(i) (4 marks) Sketch the traces of S in the $z=1$ and $y=0$ planes.

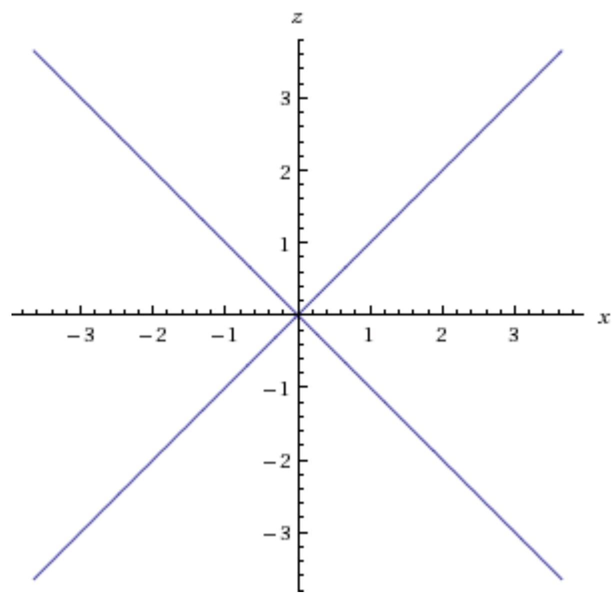
Solution:

- For $z=1$: The equation for the trace is $1=x^2+4y^2$, which is equivalent to $x^2+\frac{y^2}{\frac{1}{4}}=1$. Thus, in the xy-plane, the graph of the trace is an ellipse centered at $(0,0)$ with x-intercepts $(1,0)$ and

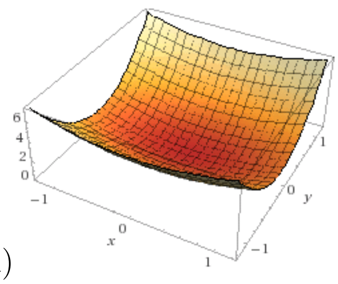
$(-1, 0)$, and y-intercepts $\left(0, \frac{1}{2}\right)$ and $\left(0, -\frac{1}{2}\right)$.



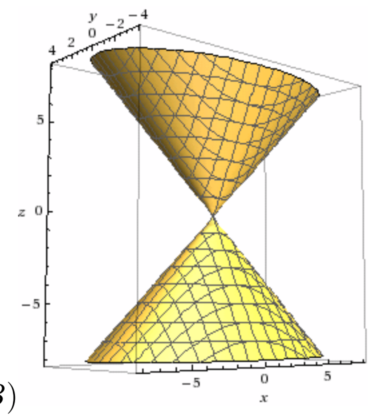
- For $y = 0$: The equation for the trace is $z^2 = x^2$, which is equivalent to $z = \pm x$. Thus, in the xz-plane, the graph of the trace consists of two lines $z = x$ and $z = -x$.



(ii) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?



(A)



(B)

Solution: The answer is (B) since in (A), the trace at $y = 0$ looks like a parabola, not two intersecting lines.

2. (10 marks) Let R be the semicircular region $\{x^2 + y^2 \leq 4, x \leq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 + 2x$$

on the *boundary of the region R*.

Solution: The boundary of the region R consists of two pieces: the semicircular arc which can be parametrized by $x = 2\cos\theta$ and $y = 2\sin\theta$ for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, and the vertical segment $x = 0$ for $-2 \leq y \leq 2$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the semicircular arc*: We have that $f(x, y) = g(\theta) = (2\cos\theta)^2 + (2\sin\theta)^2 + 2(2\cos\theta) = 4 + 4\cos\theta$ for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Then, $g'(\theta) = -4\sin\theta = 0$ if and only if $\theta = \pi$. So, there are 3 points where extrema can occur: $(-2, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).

- *On the vertical segment*: We have that $f(0, y) = h(y) = y^2$ for $-2 \leq y \leq 2$. So, $h'(y) = 2y = 0$ when $y = 0$. So, there are 3 points where extrema can occur: $(0, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).

Evaluate f at those points, we get:

$$f(-2, 0) = 0, \quad f(0, -2) = 4, \quad f(0, 2) = 4, \quad f(0, 0) = 0$$

Thus, on the boundary of R, f attains the absolute maximum value 4 at the points $(0, -2)$ and $(0, 2)$ and the absolute minimum value 0 at the points $(-2, 0)$ and $(0, 0)$.

3. (10 marks) Find *all* critical points of the following function:

$$f(x, y) = x^2y - 2xy^2 + 3xy + 4$$

Classify each point as a local minimum, local maximum, or saddle point.

Solution: Compute the first-order partial derivatives of f:

$$f_x(x, y) = 2xy - 2y^2 + 3y = y(2x - 2y + 3) \qquad f_y(x, y) = x^2 - 4xy + 3x = x(x - 4y + 3)$$

Since both f_x and f_y are defined at every point in $\mathbb{R}^2$, the only critical points of f are those at which $f_x = f_y = 0$. If $f_x = 0$, then either $y = 0$ or $2x - 2y + 3 = 0$. Consider two cases:

- If $y = 0$: Substitute $y = 0$ into f_y, we get:

  $$x(x + 3) = 0 \Rightarrow x = 0, x = -3$$

  So, we get two critical points $(0, 0)$ and $(-3, 0)$.

- If $2x - 2y + 3 = 0$: Solving for y, we get $y = x + \frac{3}{2}$. Replacing that into $f_y = 0$ and solving for x, we get:

  $$x(x - 4(x + \frac{3}{2}) + 3) = x(-3x - 3) = 0 \Rightarrow x = 0, x = -1.$$

  So, we get two critical points $(0, 3/2)$ and $(-1, 1/2)$.

Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = 2y, \quad f_{yy} = -4x, \quad f_{xy} = 2x - 4y + 3, \quad D(x, y) = -8xy - (2x - 4y + 3)^2$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(0, 0)$, $D(0, 0) = -9 < 0$, so $(0, 0)$ is a saddle point.
- At the point $(-3, 0)$, $D(-3, 0) = -9 < 0$, so $(-3, 0)$ is a saddle point.
- At the point $(0, 3/2)$, $D(0, 3/2) = -9 < 0$, so $(0, 3/2)$ is a saddle point.
- At the point $(-1, 1/2)$, $D(-1, 1/2) = 3 > 0$, and $f_{xx}(-1, 1/2) = 1 > 0$, so $(-1, 1/2)$ is a local minimum.

4. (10 marks) A consumer has the following utility function for the two types of chocolates:

$$f(x, y) = 5x^2 + 6xy + y^2 + 38x + 18y$$

where x and y represent the number of grams of milk chocolate and dark chocolate, respectively. Suppose that a gram of milk chocolate costs \$10 and a gram of dark chocolate costs \$5, and the consumer can spend \$40 on the two types of chocolates. Use Lagrange multipliers to find how much of each type of chocolates the consumer should buy to maximize his utility value. Clearly state the objective function and the constraint. *You are not required to justify that the solution you obtained is the absolute maximum.* **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

Solution: Since the consumer can spend \$40 and a gram of milk chocolate costs \$10 while a gram of dark chocolate costs \$5, we get the constraint function $g(x, y) = 10x + 5y - 40 = 0$. The objective function to maximize is the utility function $f(x, y) = 5x^2 + 6xy + y^2 + 38x + 18y$. Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned} \nabla f(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= 0 \end{aligned}$$

More explicitly, we need to solve:

$$\begin{aligned} 10x + 6y + 38 &= 10\lambda \\ 6x + 2y + 18 &= 5\lambda \\ 10x + 5y - 40 &= 0 \end{aligned}$$

Subtracting twice of the second equations from the first, we get $-2x + 2y + 2 = 0$. So, $x = y + 1$. Substituting that into the third equation, we get:

$$\begin{aligned} 10(y + 1) + 5y - 40 &= 0 \\ 15y &= 30 \\ y &= 2. \end{aligned}$$

Thus, $x = y + 1 = 3$ and $\lambda = \dfrac{6x + 2y + 18}{5} = 8$. Therefore, if the consumer buys 3 grams of milk chocolate and 2 grams of dark chocolate, then he would maximize his utility value.